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On the Propagation and Decay of Gravitational Waves*

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I. INTRODUCTION AND SUMMARY OF RESULTS

Let us denote the metric differential form of a Riemann-Einstein space by

$$ds^2 = h_{ij}(x)\, dx^i\, dx^j, \tag{1.1}$$

where the symmetric coefficients h_{ij} are functions of the variables x^0, x^1, x^2, x^3 and there is a summation on the indices i and j over the range 0, 1, 2, 3. The metric is said to be *statical* [1] if a coordinate system S can be chosen such that the above form can be written

$$ds^2 = V^2(x)\, dt^2 - g_{ij}(x)\, dx^i\, dx^j, \tag{1.2}$$

in which the g's are the coefficients of a positive definite quadratic form and the quantities V and g_{ij} are functions of the variables x^1, x^2, x^3 alone; it is to be understood of course that the second set of terms in the right member of (1.2) involves a summation over the values 1, 2, 3. It is natural to interpret the coordinate x^0 as the time variable t in the system S, as indicated in (1.2), and to consider the coordinates x^1, x^2, x^3 as the coordinates of a three dimensional Riemann space R whose fundamental metric form is given by

$$d\sigma^2 = g_{ij}\, dx^i\, dx^j. \tag{1.3}$$

This point of view will be adopted in the following discussion. For later reference we note that the general form (1.1) reduces to the statical form (1.2) when

$$h_{00} = V^2; \qquad h_{0j} = 0; \qquad h_{ij} = -g_{ij}, \qquad (i, j = 1, 2, 3). \tag{1.4}$$

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In this paper we shall treat the propagation of weak gravitational waves into the statical gravitational field defined by (1.2). Viewing the gravitational wave surface as a two dimensional surface $\Sigma(t)$, propagated in the above Riemann space R, we have constructed the compatibility conditions of the problem in Sections III and IV under the assumption that the components h_{ij} and their first derivatives are continuous across the surface $\Sigma(t)$; as so considered the problem is analogous to the well known problem of the propagation of a sound wave into a gas at rest. We shall be concerned primarily with the propagation of *spatial* gravitational waves, i.e. with wave surfaces $\Sigma(t)$ which admit discontinuities in the second partial derivatives of the purely spatial components g_{ij} appearing in the form (1.2) while the second partial derivative of the *temporal* components as represented by the coefficients h_{0j} for $j = 0, 1, 2, 3$ in (1.1) are continuous across $\Sigma(t)$. It is shown in Section V that the normal velocity of such waves is given by the value of the quantity V in (1.2) at points of the wave surface; in Section VI it is shown that the normal trajectories to the wave surfaces $\Sigma(t)$ are geodesics of zero length in the four dimensional space-time continuum whose element of distance ds is given by the form (1.2). In the special case for which the expression (1.2) for ds^2 can be approximated by the expression for the element of distance in the special theory of relativity, i.e.

$$ds^2 = c^2\,dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{1.5}$$

the wave surfaces $\Sigma(t)$ will be propagated with the constant velocity of light c and the successive positions of these surfaces will form a family of parallel surfaces having as their normal trajectories a congruence of straight lines in the three dimensional Euclidean space of the special theory of relativity.

To investigate the variation of the wave strength, defined in Section V, during the propagation of the wave we have constructed the compatibility conditions of the third order in Section VIII; in this consideration we have used the special form (1.5) rather than the general form (1.2) for the element of distance of the statical field in order to obtain the greatest possible simplification in our relations. On the basis of these compatibility conditions the differential equation for the variation of the wave strength is derived in Section IX and is found to be the same as the equation for the variation of the strength of a longitudinal or shear wave propagated in an unstrained elastic solid. Thus we find that ordinarily the strength of the wave will decrease to zero monotonically as the wave progresses although in certain cases, depending on the mean curvature of the wave front, the strength will approach infinity over some surface which will, presumably, revert to a wave bearing a finite but stronger discontinuity.

The velocity of a temporal wave or discontinuity, defined in Section VII, does not have a unique determination and in this important aspect differs from the case of the spatial wave. This suggests that such discontinuities are associated with the presence of material bodies rather than with waves in free space. Further investigation of purely temporal discontinuities, as well as the investigation of discontinuities of mixed temporal and spatial character, would appear to be of interest; one might also consider stronger gravitational waves, including gravitational shocks, which will produce a discontinuous or abrupt change in the gravitational force. It would seem likely that such waves or discontinuities will be found to be of importance in the relativistic theory of atomic phenomena.

II. Auxiliary Formulas

We shall collect certain formulas in this section for convenient reference in the following discussion [2]. Let us first observe that in the region in front of the gravitational wave $\Sigma(t)$, where the form (1.2) for the element of distance ds is assumed to be applicable, we have

$$h^{00} = \frac{1}{V^2}; \qquad h^{0j} = 0; \qquad h^{ij} = -g^{ij}, \qquad (i, j = 1, 2, 3), \tag{2.1}$$

where the h^{ij} for $i, j = 0, 1, 2, 3$ and the g^{ij} for $i, j = 1, 2, 3$ are the contravariant components of the metric tensors h and g whose covariant components occur in (1.1) and (1.2) respectively; these relations follow as a direct consequence of (1.4). Also the components Λ^i_{jk} of the affine connection, which are given in general by the formula

$$\Lambda^i_{jk} = \frac{1}{2} h^{im} \left(\frac{\partial h_{mj}}{\partial x^k} + \frac{\partial h_{mk}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^m} \right), \tag{2.2}$$

for $i, j, k, m = 0, 1, 2, 3$ have the following special values in the region in front of the wave $\Sigma(t)$, namely

$$\begin{aligned} &\Lambda^i_{jk} = \Gamma^i_{jk}; \qquad \Lambda^0_{jk} = 0; \qquad \Lambda^i_{0k} = 0, \\ &\Lambda^0_{0k} = \frac{\partial \log V}{\partial x^k}; \qquad \Lambda^i_{00} = V g^{im} \frac{\partial V}{\partial x^m}; \qquad \Lambda^0_{00} = 0, \end{aligned} \tag{2.3}$$

in which the indices i, j, k, m have the values $1, 2, 3$; the Γ's in these relations are the components of the affine connection for the Riemann space R whose element of distance $d\sigma$ is given by (1.3), i.e.

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right). \tag{2.4}$$

Let us now represent the moving surface $\Sigma(t)$ by the parametric equations

$$x^i = x^i(u^1, u^2, t), \qquad (i = 1, 2, 3), \tag{2.5}$$

and let us denote by G the velocity of propagation of this surface in the direction of its unit normal ν. We assume that ν is directed into the space R; hence $G > 0$ and since ν is a unit vector in R by hypothesis we have

$$g_{ij}\, \nu^i \nu^j = 1, \tag{2.6}$$

where the ν^i are the contravariant components of ν. It will furthermore be assumed that the functions $x^i(u, t)$ are continuously differentiable and that the surface defined by (2.5) is regular in the sense that the functional matrix $||\partial x^i/\partial u^\alpha||$ is everywhere different from zero. Then the partial derivatives $\partial x^i/\partial u^\alpha$, which we shall denote by $x_\alpha{}^i$ for brevity, are the components of a contravariant vector in the space R, tangent to the surface $\Sigma(t)$, for α fixed, and the components of a covariant vector on the surface $\Sigma(t)$ for i fixed. Hence we can write

$$g_{ij}\, x_\alpha{}^i \nu^j = 0. \tag{2.7}$$

Also the covariant components of the fundamental metric tensor of the surface $\Sigma(t)$ are given by the equations

$$g_{\alpha\beta} = g_{ij}\, x_\alpha{}^i x_\beta{}^j, \tag{2.8}$$

where the indices α, β have the values 1, 2 and there is a summation on the indices i, j over the range 1, 2, 3. We shall likewise make use of the well known formula

$$g^{\alpha\beta}\, x_\alpha{}^i x_\beta{}^j = g^{ij} - \nu^i \nu^j, \tag{2.9}$$

in which the $g^{\alpha\beta}$ are the contravariant components of the metric tensor of the surface $\Sigma(t)$ and the indices α, β are summed over the range 1, 2 in accordance with the usual convention. In the following discussion Greek indices will be restricted to the values 1 and 2, as in the equations (2.8) and (2.9), and will appear on the symbols of quantities intrinsically associated with the two dimensional surface $\Sigma(t)$. Latin indices, on the other hand, may have the range 0, 1, 2, 3 or the range 1, 2, 3; the range of such indices will be stated when not sufficiently clear from the context. Some use will also be made of the equations

$$\nu_{i,\alpha} = -\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{i\tau}; \qquad x_{i\tau} = g_{ik}\, x_\tau{}^k, \tag{2.10}$$

where the comma denotes covariant differentiation of the space vector ν based on the metric of the surface $\Sigma(t)$; moreover the quantities $b_{\alpha\sigma}$ are the coefficients of the second fundamental form of the surface $\Sigma(t)$ and the $x_{i\tau}$ are defined by the second set of these equations. We shall also have occasion to employ the mean curvature Ω of the surface $\Sigma(t)$ which is defined by the equation

$$\Omega = \tfrac{1}{2} g^{\sigma\tau} b_{\sigma\tau}. \tag{2.11}$$

The relations

$$\frac{\delta x^i}{\delta t} = G\nu^i\,; \qquad G = \frac{d\sigma}{dt}, \tag{2.12}$$

are obviously valid, where $\delta x^i/\delta t$ is the δ time derivative [3] of the coordinate x^i (for $i = 1, 2, 3$) and σ is the arc length along the normal trajectories to the surfaces $\Sigma(t)$ when the trajectories are considered as curves in the Riemann space R. Finally we wish to call attention to the equations [3]

$$\frac{D\nu^i}{Dt} = -\, g^{\alpha\beta} G_{,\alpha}\, x_\beta{}^i, \tag{2.13}$$

where the quantities $D\nu^i/Dt$ are the components of the absolute time derivative of the vector ν along the normal trajectories and the $G_{,\alpha}$ are the partial derivatives of the velocity G, considered as a scalar on the surface $\Sigma(t)$, with respect to the surface coordinates u^α; the components of the above time derivative are given by the formula

$$\frac{D\nu^i}{Dt} = \frac{d\nu^i}{dt} + G\Gamma^i_{jk}\, \nu^j\, \nu^k. \tag{2.14}$$

III. Geometrical and Kinematical Compatibility Conditions of the Second Order

We make the following assumptions:

A_1. *The components h_{ij} for $i, j = 0, 1, 2, 3$ are continuous over the surface $\Sigma(t)$,*

A_2. *The first partial derivatives of the components h_{ij} for $i, j, = 0, 1, 2, 3$ with respect to the coordinates x^1, x^2, x^3 are continuous over the surface $\Sigma(t)$.*

Using a bracket, as customary, to denote the difference in the values of a quantity on the two sides of the surface $\Sigma(t)$, the above assumptions can be expressed by writing

$$[h_{ij}] = 0, \qquad (i, j = 0, 1, 2, 3), \tag{3.1}$$

$$[h_{ij/k}] = 0, \qquad (i, j = 0, 1, 2, 3; \quad k = 1, 2, 3), \tag{3.2}$$

where the symbol / is employed to denote partial differentiation with respect to the coordinates x^k; when followed by a Greek letter in the following discussion this symbol will indicate partial differentiation with respect to the curvilinear coordinates u^α of the surface $\Sigma(t)$. But from (2.12) and (3.1) we have

$$\frac{\delta [h_{ij}]}{\delta t} = [h_{ij/k}]\, G\nu^k + \left[\frac{\partial h_{ij}}{\partial t}\right] = 0,$$

in which the index k is summed over the values 1, 2, 3; hence from (3.2) it follows that

$$\left[\frac{\partial h_{ij}}{\partial t}\right] = 0. \tag{3.3}$$

Identifying x^0 with the time variable t, the following result can therefore be stated. *The relations* (3.2) *for* $k = 0$ *are also valid.*

Now consider the identities

$$[h_{ij/k}]_{/\alpha} = [h_{ij/km}]\, x_\alpha{}^m = 0, \tag{3.4}$$

in which $i, j = 0, 1, 2, 3$; $k, m = 1, 2, 3$ and the index m is summed over its range of values. Multiplying (3.4) by $g^{\alpha\beta}\, x_\beta{}^n$ and summing on repeated indices as indicated we find that

$$[h_{ij/km}]\, (g^{mn} - \nu^m \nu^n) = 0,$$

when use is made of (2.9). When these relations are solved for the brackets $[h_{ij/km}]$ the resulting equations can be written in the form

$$[h_{ij/km}] = B_{ijk}\, \nu_m, \tag{3.5}$$

where the B_{ijk} are quantities defined on the surface $\Sigma(t)$. But the right members of (3.5) must be symmetric in the indices k and m since the left members of these equations have this property; hence

$$B_{ijk}\, \nu_m = B_{ijm}\, \nu_k. \tag{3.6}$$

Multiplying (3.6) by ν^m and summing on the repeated index m, we obtain

$$B_{ijk} = C_{ij}\, \nu_k, \tag{3.7}$$

where the C's are quantities defined on the surface $\Sigma(t)$ and are in fact given by

$$C_{ij} = B_{ijk}\, \nu^k = [h_{ij/km}]\, \nu^k \nu^m.$$

Finally when we substitute the values of the B_{ijk} given by (3.7) into the equations (3.5) we have

$$[h_{ij/km}] = C_{ij}\, \nu_k\, \nu_m, \qquad (i, j = 0, 1, 2, 3; \quad k = 1, 2, 3). \tag{3.8}$$

The relations (3.8) are the *geometrical conditions of compatibility of the second order* for the components h_{ij}. To obtain the corresponding kinematical conditions of compatibility we consider the following two sets of identities, namely

$$\frac{\delta}{\delta t}[h_{ij/k}] = [h_{ij/km}]\, G\nu^m + \left[\frac{\partial^2 h_{ij}}{\partial x^k\, \partial t}\right] = 0, \tag{3.9}$$

$$\frac{\delta}{\delta t}\left[\frac{\partial h_{ij}}{\partial t}\right] = \left[\frac{\partial^2 h_{ij}}{\partial x^k\, \partial t}\right] G\nu^k + \left[\frac{\partial^2 h_{ij}}{\partial t^2}\right] = 0, \tag{3.10}$$

from which we immediately obtain

$$\left[\frac{\partial^2 h_{ij}}{\partial x^k\, \partial t}\right] = -\,GC_{ij}\, \nu_k, \qquad (i, j = 0, 1, 2, 3; \quad k = 1, 2, 3), \tag{3.11}$$

$$\left[\frac{\partial^2 h_{ij}}{\partial t^2}\right] = G^2\, C_{ij}, \qquad (i, j = 0, 1, 2, 3). \tag{3.12}$$

In fact the relations (3.11) result from (3.8) and (3.9); similarly we obtain (3.12) from (3.10) and (3.11). *The relations* (3.11) *and* (3.12) *are the kinematical conditions of compatibility of the second order* for the problem under consideration.

Let us now define quantities λ_i for $i = 0, 1, 2, 3$ by the equations

$$\lambda_0 = G; \qquad \lambda_k = -\,\nu_k, \qquad (k = 1, 2, 3). \tag{3.13}$$

Also defining the quantities λ^i by the expressions $h^{ij}\lambda_j$ in the usual manner, we have

$$\begin{aligned} \lambda^0 &= h^{00}\,\lambda_0 = G/V^2, \\ \lambda^i &= -\,g^{ij}\,\lambda_j = \nu^i, \end{aligned} \tag{3.14}$$

for $i = 1, 2, 3$ when account is taken of (2.1) and (3.13). The equations (3.14) will be used in the following Section V; however we can make immediate use of (3.13) to combine the above compatibility conditions (3.8), (3.11) and (3.12) into the single set of relations

$$[h_{ij/km}] = C_{ij}\, \lambda_k\, \lambda_m, \qquad (i, j, k, m = 0, 1, 2, 3). \tag{3.15}$$

It is readily observed that the statical form (1.2) of the element of distance ds is invariant under time and coordinate transformations of the type

$$t = a\bar{t} + b; \qquad x^i = \phi^i(\bar{x}^1, \bar{x}^2, \bar{x}^3), \qquad (i = 1, 2, 3), \tag{3.16}$$

where a and b are constants, with $a \neq 0$, and the functions $\phi^i(\bar{x})$ define a differentiable transformation $x \leftrightarrow \bar{x}$ of the coordinates of the Riemann space R; the first of the equations (3.16) allows, of course, the possibility of the usual change of origin and unit of time. Also under such transformations the coefficient h_{00} in the form (1.1), or the coefficient V^2 in (1.2), will be the component of a scalar S; the coefficients h_{0j} for $j = 1, 2, 3$ (which do not necessarily vanish in the region behind the wave surface Σ) will be the components of a covariant vector J; moreover we easily see that the quantities h_{ij} for $i, j = 1, 2, 3$ can be regarded as the components of a tensor T under transformations (3.16). Hence the equations (3.13) and (3.14) will be invariant and the quantities λ_i and λ^i, given by these equations, will be the components of a covariant and a contravariant vector respectively under such time and coordinate transformations. With regard to the quantities $[h_{ij/km}]$ the following result can be stated. *The quantities* $[h_{ij/km}]$ *for* $i, j, k, m = 0, 1, 2, 3$ *transform as the components of a covariant tensor when the time and space coordinates undergo a transformation* (3.16). To see this we have merely to differentiate the equations of transformation of the components h_{ij} successively with respect to x^k and x^m and then subtract corresponding members of the resulting equations after evaluation on the two sides of the surface $\Sigma(t)$; the result in question will then follow as a consequence of the fact that the quantities $[h_{ij}]$ and $[h_{ij/k}]$ vanish on the surface $\Sigma(t)$ from the assumptions A_1 and A_2. Finally it is seen that the symmetric quantities C_{ij} in the equations (3.15) can be considered to be the components of a tensor under transformations (3.16) because of the tensor character of the other quantities, i.e. the components λ_i and $[h_{ij/km}]$, in these equations.

IV. The Dynamical Conditions of Compatibility

By definition the components B^i_{jkm} of the curvature tensor of the Riemann-Einstein space are given by equations of the form

$$B^i_{jkm} = \frac{\partial \Lambda^i_{jk}}{\partial x^m} - \frac{\partial \Lambda^i_{jm}}{\partial x^k} + \ldots, \tag{4.1}$$

where the Λ's are the components of the affine connection of this space and the dots denote terms which are quadratic in the components Λ;

the explicit form of these quadratic terms will not be needed in the discussion since the Λ^i_{jk} are continuous on the wave surface $\Sigma(t)$ from the assumptions in Section III. It follows immediately from (4.1) that

$$[B^i_{jkm}] = \left[\frac{\partial \Lambda^i_{jk}}{\partial x^m}\right] - \left[\frac{\partial \Lambda^i_{jm}}{\partial x^k}\right]. \tag{4.2}$$

Now when account is taken of the equations (2.2) giving the quantities Λ^i_{jk} we have

$$\left[\frac{\partial \Lambda^i_{jk}}{\partial x^m}\right] = \tfrac{1}{2} h^{ie} \left([h_{ej/km}] + [h_{ek/jm}] - [h_{jk/em}]\right), \tag{4.3}$$

in which all indices can be considered to have the full range of values 0, 1, 2, 3. Making the substitutions (4.3) and using the relations (3.15) we now find that the equations (4.2) can be written

$$[B^i_{jkm}] = \tfrac{1}{2} h^{ie} (C_{ek}\, \lambda_j\, \lambda_m - C_{jk}\, \lambda_e\, \lambda_m - C_{em}\, \lambda_j\, \lambda_k + C_{jm}\, \lambda_e\, \lambda_k). \tag{4.4}$$

It is assumed in this discussion that the wave surface $\Sigma(t)$ is propagated in *free space,* i.e. in a Riemann-Einstein space subject to the field equations

$$B_{jk} = B^n_{jkn} = 0, \qquad (j, k, n = 0, 1, 2, 3). \tag{4.5}$$

Contracting the indices i and m in the equations (4.4) we thus have

$$h^{em} (C_{ek}\, \lambda_j\, \lambda_m - C_{jk}\, \lambda_e\, \lambda_m - C_{em}\, \lambda_j\, \lambda_k + C_{jm}\, \lambda_e\, \lambda_k) = 0. \tag{4.6}$$

The relations (4.6), which can be considered to suppliment the compatibility conditions (3.15) are the *dynamical conditions* of compatibility of the problem. These conditions can also be written as

$$(C_{km}\, \lambda^m)\, \lambda_j + (C_{jm}\, \lambda^m)\lambda_k - (C_{em}\, h^{em})\lambda_j\, \lambda_k - (\lambda_m\, \lambda^m) C_{jk} = 0, \tag{4.7}$$

in which the full range of values 0, 1, 2, 3 is assumed by all indices.

V. Spatial Waves and Wave Strength

Certain additional restrictions beyond those contained in the assumptions A_1 and A_2 in Section III will now be imposed on the discontinuities over the wave surface $\Sigma(t)$; these restrictions will be stated in terms of the quantities S, J and T introduced at the end of Section III.

B_1. *The partial coordinate derivatives of the second order of the components of the scalar S and vector J are continuous across the surface $\Sigma(t)$.*

B_2. *There is a discontinuity over the surface $\Sigma(t)$ in the partial coordinate derivatives of the second order of the components of the tensor T.*

It follows from assumption B_1 that all second order derivatives of the components of the scalar S and vector J are continuous over $\Sigma(t)$, i.e. time derivatives and mixed time and coordinate derivatives of the second order are also continuous over this surface (see corresponding result in Section III). A wave surface $\Sigma(t)$ over which the conditions B_1 and B_2 are satisfied will be referred to as a *spatial wave* since the relevant discontinuities over this surface are limited to discontinuities in the components of the metric tensor of the Riemann space R. Now from (3.15) we have

$$[h_{0j/km}] = C_{0j}\,\lambda_k\,\lambda_m = 0, \tag{5.1}$$

for $j, k, m = 0, 1, 2, 3$. But from (5.1) we see that

$$C_{0j} = 0, \qquad (j = 0, 1, 2, 3). \tag{5.2}$$

Using the relations (5.2), the equations (4.7) become

$$(C_{ak}\,\nu^a)\lambda_j + (C_{aj}\,\nu^a)\lambda_k + (C_{ab}\,g^{ab})\lambda_j\,\lambda_k - (\lambda_m\,\lambda^m)C_{jk} = 0, \tag{5.3}$$

where $a, b = 1, 2, 3$; $j, k, m = 0, 1, 2, 3$. Restricting the indices j, k in (5.3) to the values $1, 2, 3$ we now readily deduce

$$(1 + \lambda_m\,\lambda^m)C_{aj}\,\nu^a = (C_{ab}\,g^{ab})\nu_j - (C_{ab}\,\nu^a\,\nu^b)\nu_j, \tag{5.4}$$

$$(2 + \lambda_m\,\lambda^m)C_{ab}\,\nu^a\,\nu^b = C_{ab}\,g^{ab}. \tag{5.5}$$

In fact the relations (5.4) are obtained from (5.3) by multiplication by ν^k while (5.5) results from (5.4) when we multiply by ν^j and sum on the repeated indices in each case as required. Also if we multiply (5.3) by the quantities g^{jk} and sum on the repeated indices we find that the resulting equation can be given the form

$$(1 - \lambda_m\,\lambda^m)C_{ab}\,g^{ab} = 2C_{ab}\,\nu^a\,\nu^b. \tag{5.6}$$

Finally, adding corresponding members of (5.5) and (5.6), we obtain

$$\lambda_m\,\lambda^m\,(C_{ab}\,\nu^a\,\nu^b - C_{ab}\,g^{ab}) = 0. \tag{5.7}$$

Now assume $\lambda_m\,\lambda^m \neq 0$. Then from (5.7) we have

$$C_{ab}\,\nu^a\,\nu^b = C_{ab}\,g^{ab}, \tag{5.8}$$

and hence (5.6) becomes

$$(1 + \lambda_m\,\lambda^m)C_{ab}\,g^{ab} = 0. \tag{5.9}$$

But the parenthesis expression in (5.9) is seen to have the value G^2/V^2 when use is made of the relations (3.13) and (3.14). Since G does not vanish and V is not infinite by hypothesis it therefore follows from (5.9) that

$$C_{ab}\, g^{ab} = 0. \tag{5.10}$$

Hence we have

$$C_{ab}\, \nu^a \nu^b = 0, \tag{5.11}$$

from (5.8). But from (5.10) and (5.11) we see that the right members of (5.4) must vanish and, since the parenthesis expression in the left members of these equations has been observed to be different from zero, we conclude that

$$C_{aj}\, \nu^a = 0, \qquad (j = 1, 2, 3). \tag{5.12}$$

From the relations (5.10) and (5.12) it now follows that (5.3) reduces to

$$(\lambda_m \lambda^m) C_{jk} = 0, \qquad (j, k = 1, 2, 3),$$

and hence the quantities C_{jk} in these equations must vanish since $\lambda_m \lambda^m$ is different from zero by hypothesis. But this result is in contradiction with the above assumption B_2. Hence

$$\lambda_m \lambda^m = \frac{G^2}{V^2} - 1 = 0. \tag{5.13}$$

In other words the normal velocity G has the value V at points of the spatial wave surface $\Sigma(t)$.

In view of the relation (5.13) the above equations (5.3) now reduce to

$$(C_{ak}\, \nu^a)\lambda_j + (C_{aj}\, \nu^a)\lambda_k + (C_{ab}\, g^{ab})\lambda_j \lambda_k = 0, \tag{5.14}$$

with $a, b = 1, 2, 3$; $j, k = 0, 1, 2, 3$. Taking $j = k = 0$ in (5.14), using (5.2) and the relation $\lambda_0 = G$ from (3.13), with $G \neq 0$ by hypothesis, we have

$$C_{ab}\, g^{ab} = 0. \tag{5.15}$$

Hence (5.14) becomes

$$(C_{ak}\, \nu^a)\lambda_j + (C_{aj}\, \nu^a)\lambda_k = 0. \tag{5.16}$$

Now put $j = 0$ and let k have the values $1, 2, 3$ in (5.16); this gives

$$C_{ak}\, \nu^a = 0, \qquad (k = 1, 2, 3). \tag{5.17}$$

The equations (5.15) and (5.17), as well as the above equations (5.2) will be found to have a useful application in the following discussion.

The magnitude of the discontinuity over the wave surface $\Sigma(t)$ is measured in general by the quantities C_{ij} for $i, j = 0, 1, 2, 3$ in accordance with the relations (3.15). In the case of the spatial wave for which $C_{0j} = 0$ for $j = 0, 1, 2, 3$ it is therefore natural to define the wave strength W by the equation

$$W = \sqrt{g^{ij} g^{km} C_{ik} C_{jm}}, \tag{5.18}$$

from which it follows that $W = 0$ implies $C_{ij} = 0$ for $i, j = 1, 2, 3$ and conversely. In the final section of this paper we shall be concerned with the variation of the strength W of a spatial wave during its propagation.

VI. Wave Trajectories as Geodesics of Zero Length

Consider the normal trajectories N to the family of surfaces given by the successive positions of the spatial waves $\Sigma(t)$ in the Riemann space R. Denoting by σ the arc length along any curve N, based on the metric of R, and by s its arc length as a curve in the four dimensional space-time continuum whose element of distance ds is given by (1.2), we have

$$ds^2 = \left\{ V^2 \left(\frac{dt}{d\sigma}\right)^2 - g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \right\} d\sigma^2 = \left(\frac{V^2}{G^2} - 1\right) d\sigma^2 = 0,$$

from (5.13), i.e. the trajectories N are *curves of zero length* in the four dimensional continuum.

More definitive information regarding the trajectories N can be obtained from the equations (2.13) which can be shown to yield the relations

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} + V g^{ij} \frac{\partial V}{\partial x^j} = \frac{2}{V} \left(\frac{\partial V}{\partial x^j} \frac{dx^j}{dt}\right) \frac{dx^i}{dt}, \tag{6.1}$$

along these curves when we take account of the fact that $G = V$, as shown in Section V, make the substitutions

$$\nu^i = \frac{dx^i}{d\sigma} = \frac{1}{V} \frac{dx^i}{dt},$$

and also use the formula (2.14) and the identities (2.9); the details of the derivation of (6.1) will not be carried out here but will be left to the reader. Introducing the components Λ given by (2.3), the equations (6.1) can be written

$$\frac{d^2x^i}{dt^2} + \Lambda^i_{km} \frac{dx^k}{dt} \frac{dx^m}{dt} = \frac{2}{V} \left(\frac{\partial V}{\partial x^j} \frac{dx^j}{dt}\right) \frac{dx^i}{dt}, \tag{6.2}$$

where $i, j = 1, 2, 3$ and there is a summation on the indices k, m over the range $0, 1, 2, 3$. To simplify the equations further let us make a differentiable parameter transformation $t \leftrightarrow \tau$ in consequence of which the equations (6.2) become

$$\left\{\frac{d^2x^i}{d\tau^2} + \Lambda^i_{km}\frac{dx^k}{d\tau}\frac{dx^m}{d\tau}\right\}\left(\frac{d\tau}{dt}\right)^2 + \left\{\frac{d^2\tau}{dt^2} - \frac{2}{V}\frac{\partial V}{\partial x^j}\frac{dx^j}{d\tau}\left(\frac{d\tau}{dt}\right)^2\right\}\frac{dx^i}{d\tau} = 0.$$

Now put

$$\frac{d^2\tau}{dt^2} = \frac{2}{V}\frac{\partial V}{\partial x^j}\frac{dx^j}{d\tau}\left(\frac{d\tau}{dt}\right)^2 = \frac{2}{V}\frac{dV}{dt}\frac{d\tau}{dt},$$

and integrate this equation along the trajectory N in question to obtain

$$\tau = A\int V^2\,dt + B, \tag{6.3}$$

where A and B are arbitrary constants. Hence when we make the parameter transformation (6.3), with $A \neq 0$, the above equations (6.2) assume the simpler form

$$\frac{d^2x^i}{d\tau^2} + \Lambda^i_{km}\frac{dx^k}{d\tau}\frac{dx^m}{d\tau} = 0. \tag{6.4}$$

We have shown that the equations (6.4) are satisfied along the normal trajectories N when the index i is restricted to the values $1, 2, 3$ and the indices k, m are summed over the range $0, 1, 2, 3$. But, taking $i = 0$, it is readily seen that the left member of (6.4) vanishes identically along the curves N. It is customary to express a result of this type by saying that *the wave trajectories N are geodesics of zero length* in the space-time continuum whose metric is defined by (1.2); this terminology is employed since the equations (6.4), in which the indices i, k, m have the full range of values from zero to three, are formally equivalent to the equations of the geodesics in a Riemann space.

VII. Problem of Temporal Discontinuities

The question will now be raised as to whether wave surfaces $\Sigma(t)$ can exist over which there will be a discontinuity in the second partial derivatives of the components h_{0j} for $j = 0, 1, 2, 3$ while the corresponding partial derivatives of the components h_{ij} for $i, j = 1, 2, 3$ will be continuous over these surfaces. Such waves, or discontinuities, will be called *temporal* to distinguish them from the spatial waves discussed in Section V; they will be characterized by the previous assumptions A_1 and A_2 together with the following assumptions C_1 and C_2 stated in terms of the quantities S, J and T defined in Section III.

C_1. *There is a discontinuity over the surface $\Sigma(t)$ among the partial coordinate derivatives of the second order of the components of the scalar S and the vector J,*

C_2. *The partial coordinate derivatives of the second order of the components of the tensor T are continuous over the surface $\Sigma(t)$.*

We see immediately from the relations (3.15) that

$$C_{ij} = 0, \qquad (i, j = 1, 2, 3), \tag{7.1}$$

as a result of the above condition C_2; hence it follows from (3.15) and (7.1) that the time and the mixed time and coordinate derivatives of the second order of the components of T are likewise continuous over the surface $\Sigma(t)$. Taking $j, k = 1, 2, 3$ and making use of the relations (3.13) and (3.14) for the quantities λ the equations (4.7) now become

$$GC_{0k}\, \nu_j + GC_{0j}\, \nu_k + C_{00}\, \nu_j\, \nu_k = 0. \tag{7.2}$$

When we multiply (7.2) by ν^k and then multiply the resulting equations by ν^j we are led to the following relations

$$G(C_{0k}\, \nu^k)\nu_j + GC_{0j} + C_{00}\, \nu_j = 0, \tag{7.3}$$

$$2G(C_{0k}\, \nu^k) + C_{00} = 0. \tag{7.4}$$

Eliminating the combination $C_{0k}\, \nu^k$ between the equations (7.3) and (7.4) we obtain

$$C_{0j} = -\left(\frac{C_{00}}{2G}\right)\nu_j, \qquad (j = 1, 2, 3). \tag{7.5}$$

Now when we substitute the values of the C_{0j} given by (7.5) into the equations (7.2) we find that the latter are satisfied identically; hence (7.5) is equivalent to the complete set of relations (7.2).

If we choose $j = k = 0$ or $j = 0$; $k = 1, 2, 3$ in (4.7) we also find that these equations are satisfied when account is taken of the relations (7.5). Hence (7.5) gives all the conditions on the C's which are obtainable from the equations (4.7) under the above assumption C_2. Thus the velocity of propagation G of a temporal discontinuity does not have the unique algebraic determination corresponding to the case of the spatial wave. *This fact suggests that such discontinuities are directly associated with the presence of material bodies rather than with the propagation of a wave in free space.* We shall not, therefore, pursue the investigation of these discontinuities in the present paper.

VIII. Compatibility Conditions of the Third Order

In the derivation of the third order compatibility conditions in this section it will be assumed that the statical field in front of the wave surface $\Sigma(t)$ can be approximated by the expression (1.5) for the element of distance ds in the special theory of relativity; this assumption will simplify the derivations considerably since the conditions in question will not be invariant under the general coordinate transformations $x \leftrightarrow \bar{x}$ which are permissible in the Riemann space R. *Furthermore under this assumption the normal velocity G of propagation of the wave surface $\Sigma(t)$ will be equal to the constant velocity c of propagation of light, the normal trajectories N will be straight lines from the equations* (2.13), *and the successive positions of the wave will form a family of parallel surfaces* [2] *in the three dimensional Euclidean space of the special theory of relativity.*

We now have

$$[h_{ij/km}]_{/\alpha} = [h_{ij/kmp}]\, x^p_\alpha, \tag{8.1}$$

where $i, j = 0, 1, 2, 3$; $k, m, p = 1, 2, 3$ and the index $\alpha = 1, 2$. It will be understood in the remainder of this section, without further mention, that the indices i, j have the range 0, 1, 2, 3 and that other Latin indices are restricted to the values 1, 2, 3; Greek indices will have the range 1, 2 as in the preceding work. We shall continue to use the symbol / to denote partial differentiation with respect to the variables in question and will reserve the comma to indicate covariant differentiation.

Multiplying (8.1) by $g^{\alpha\beta} x_\beta{}^n$ and summing on repeated indices we obtain a set of equations of the form

$$[h_{ij/kmn}] = E_{ijkm}\, \nu_n + g^{\alpha\beta}\, [h_{ij/km}]_{/\alpha}\, x_{n\beta}, \tag{8.2}$$

when use is made of the formula (2.9) and account is taken of the fact that g_{ij} and g^{ij} are equal to the corresponding Kronecker δ's from the above assumption; in writing the equations (8.2) the index n has been lowered which is permissible since there is now no distinction between the covariant and contravariant positions of Latin indices. Since the quantities $[h_{ij/kmn}]$ are symmetric in the indices m, n the right members of (8.2) must be symmetric in these indices; hence we must have

$$E_{ijkm}\, \nu_n + g^{\alpha\beta}\, [h_{ij/km}]_{/\alpha}\, x_{n\beta} = E_{ijkn}\, \nu_m + g^{\alpha\beta}\, [h_{ij/kn}]_{/\alpha}\, x_{m\beta}. \tag{8.3}$$

Multiplying (8.3) by ν_n and then multiplying the resulting equations by ν_k and in each case summing on repeated indices we obtain two sets of equations which (after some changes of indices) can be written

$$E_{ijkm} = (E_{ijkn}\, \nu_n)\nu_m + g^{\alpha\beta}\, [h_{ij/kn}]_{/\alpha}\, \nu_n\, x_{m\beta}, \tag{8.4}$$

$$E_{ijkm}\, \nu_m = E_{ij}\, \nu_k + g^{\alpha\beta}\, [h_{ij/ab}]_{/\alpha}\, \nu_a\, \nu_b\, x_{k\beta}, \tag{8.5}$$

where the E_{ij} are quantities defined on the surface $\Sigma(t)$ and are given by

$$E_{ij} = E_{ijkm} \nu_k \nu_m.$$

When the expressions for $E_{ijkn} \nu_n$ given by (8.5) are substituted into the right members of (8.4) we find that

$$\begin{aligned} E_{ijkm} = E_{ij} \nu_k \nu_m &+ g^{\alpha\beta} [h_{ij/ab}]_{/\alpha} \nu_a \nu_b \nu_m x_{k\beta} \\ &+ g^{\alpha\beta} [h_{ij/kn}]_{/\alpha} \nu_k x_{m\beta}. \end{aligned} \tag{8.6}$$

Some alteration of the form of the right members of (8.6) is advisable for our purpose. Thus we have

$$\begin{aligned} [h_{ij/kn}]_{/\alpha} \nu_n &= (C_{ij} \nu_k \nu_n)_{/\alpha} \nu_n \\ &= C_{ij/\alpha} \nu_k + C_{ij} \nu_{k/\alpha} + C_{ij} \nu_k \nu_{n/\alpha} \nu_n \\ &= C_{ij/\alpha} \nu_k - C_{ij} g^{\sigma\tau} b_{\alpha\sigma} x_{k\tau}, \end{aligned} \tag{8.7}$$

when use made of the relations (2.10) and (3.15) and account is taken of the fact that ν is a unit vector; also

$$[h_{ij/ab}]_{/\alpha} \nu_a \nu_b = (C_{ij} \nu_a \nu_b)_{/\alpha} \nu_a \nu_b = C_{ij/\alpha}. \tag{8.8}$$

Making the substitutions (8.7) and (8.8) the equations (8.6) become

$$\begin{aligned} E_{ijkm} = E_{ij} \nu_k \nu_m &+ C_{ij/\alpha} g^{\alpha\beta}(\nu_k x_{m\beta} + \nu_m x_{k\beta}) \\ &- C_{ij} g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma} x_{k\tau} x_{m\beta}. \end{aligned} \tag{8.9}$$

where the $b_{\alpha\sigma}$ are the coefficients of the second fundamental form of the surface $\Sigma(t)$; we note that the expressions in the right members of these relations are symmetric in the indices i, j and also in the indices k, m as required.

Treating the last set of terms in (8.2) in a corresponding manner we are led to the relations

$$[h_{ij/km}]_{/\alpha} = C_{ij/\alpha} \nu_k \nu_m - C_{ij} g^{\sigma\tau} b_{\alpha\sigma}(\nu_k x_{m\tau} + \nu_m x_{k\tau}). \tag{8.10}$$

Finally when we make the substitutions (8.9) and (8.10) the equations (8.2) become

$$\begin{aligned} [h_{ij/kmn}] = E_{ij} \nu_k \nu_m \nu_n &+ C_{ij/\alpha} g^{\alpha\beta}(\nu_k \nu_m x_{n\beta} + \nu_k \nu_n x_{m\beta} + \nu_m \nu_n x_{k\beta}) \\ &- C_{ij} g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma}(\nu_k x_{m\tau} x_{n\beta} + \nu_m x_{k\tau} x_{n\beta} + \nu_n x_{k\tau} x_{m\beta}). \end{aligned} \tag{8.11}$$

The relations (8.11) are the *geometrical conditions of compatibility of the third order*; we recall that the indices i, j in these equations may have the values 0, 1, 2, 3 but only the values 1, 2, 3 can be assumed by the other Latin indices.

To construct the corresponding kinematical conditions of compatibility we first consider the relations

$$\frac{\delta}{\delta t}\,[h_{ij/km}] = [h_{ij/kmn}]G\nu^n + \left[\frac{\partial^3 h_{ij}}{\partial x^k\,\partial x^m\,\partial t}\right]. \tag{8.12}$$

which are obtained when we take the δ time derivative of the quantities $[h_{ij/km}]$ and use the first set of equations (2.12). But

$$\frac{\delta}{\delta t}\,[h_{ij/km}] = \frac{\delta}{\delta t}\,(C_{ij}\,\nu_k\,\nu_m) = \frac{\delta C_{ij}}{\delta t}\,\nu_k\,\nu_m, \tag{8.13}$$

when we make the substitution (3.15) and take account of the fact that the time derivatives $\delta\nu_k/\delta t$ vanish for the case under consideration. Also we have

$$\begin{aligned}[h_{ij/kmn}]\nu^n = E_{ij}\,\nu_k\,\nu_m + C_{ij/\alpha}\,g^{\alpha\beta}(\nu_k\,x_{m\beta} + \nu_m\,x_{k\beta}) \\ - C_{ij}\,g^{\alpha\beta}\,g^{\sigma\tau}\,b_{\alpha\sigma}\,x_{k\tau}\,x_{m\beta},\end{aligned} \tag{8.14}$$

from the above conditions (8.11). Hence, making the substitutions (8.13) and (8.14) in (8.12) we arrive at the first of the kinematical conditions of compatibility, namely

$$\begin{aligned}\left[\frac{\partial^3 h_{ij}}{\partial x^k\,\partial x^m\,\partial t}\right] = -\,GE_{ij}\,\nu_k\,\nu_m - GC_{ij/\alpha}\,g^{\alpha\beta}(\nu_k\,x_{m\beta} + \nu_m\,x_{k\beta}) \\ + GC_{ij}\,g^{\alpha\beta}\,g^{\sigma\tau}\,b_{\alpha\sigma}\,x_{k\tau}\,x_{m\beta} + \frac{\delta C_{ij}}{\delta t}\,\nu_k\,\nu_m.\end{aligned} \tag{8.15}$$

The remaining kinematical conditions of compatibility are derived in a corresponding manner from the two sets of relations

$$\frac{\delta}{\delta t}\left[\frac{\partial^2 h_{ij}}{\partial x^k\,\partial t}\right] = \left[\frac{\partial^3 h_{ij}}{\partial x^k\,\partial x^m\,\partial t}\right]G\nu^m + \left[\frac{\partial^3 h_{ij}}{\partial x^k\,\partial t^2}\right],$$

$$\frac{\delta}{\delta t}\left[\frac{\partial^2 h_{ij}}{\partial t^2}\right] = \left[\frac{\partial^3 h_{ij}}{\partial x^k\,\partial t^2}\right]G\nu^k + \left[\frac{\partial^3 h_{ij}}{\partial t^3}\right].$$

We leave the details of this derivation to the reader and write down only the final form of these compatibility conditions which are given by the equations

$$\left[\frac{\partial^3 h_{ij}}{\partial x^k\,\partial t^2}\right] = G^2\,E_{ij}\,\nu_k + G^2\,C_{ij/\alpha}\,g^{\alpha\beta}\,x_{k\beta} - 2G\,\frac{\delta C_{ij}}{\delta t}\,\nu_k, \tag{8.16}$$

$$\left[\frac{\partial^3 h_{ij}}{\partial t^3}\right] = -\,G^3\,E_{ij} + 3G^2\,\frac{\delta C_{ij}}{\delta t}, \tag{8.17}$$

in which $i, j = 0, 1, 2, 3$ while the index k is restricted to the values 1, 2, 3. In addition to the compatibility conditions (8.11), (8.15), (8.16) and (8.17) certain dynamical conditions of compatibility of the third order must also be considered; this will be done in the following section in connection with the derivation of the differential equation for the variation of wave strength with which these latter conditions are closely related.

IX. Differential Equation for the Variation of Wave Strength

Differentiating the equations (4.5) with respect to the variables x^m for $m = 0, 1, 2, 3$ and evaluating on the two sides of the wave surface $\Sigma(t)$ we are led to relations of the form

$$\left[\frac{\partial^2 \Lambda^n_{jk}}{\partial x^n\, \partial x^m}\right] - \left[\frac{\partial^2 \Lambda^n_{jn}}{\partial x^k\, \partial x^m}\right] = 0. \tag{9.1}$$

Now make the substitutions (2.2) in (9.1); we thus obtain the following dynamical conditions of compatibility, namely

$$h^{np}([h_{pk/jmn}] - [h_{jk/pmn}] - [h_{pn/jkm}] + [h_{jn/pkm}]) = 0, \tag{9.2}$$

when account is taken of the continuity of the quantities h_{ij} and their first partial derivatives over the surface $\Sigma(t)$, i.e. the assumptions A_1 and A_2 in Section III. The full range of values 0, 1, 2, 3 may be assumed by all indices in the equations (9.2). It will suffice for our purpose, however, to limit our attention to the equations (9.2) for which m has the value zero; when this is done and when we furthermore take account of the relations (2.1), in which $V = G$ and the g^{ij} have the values δ^{ij}, we find that the above equations (9.2) give

$$\begin{gathered}\frac{1}{G^2}\left(\left[\frac{\partial^3 h_{0k}}{\partial x^j\, \partial t^2}\right] - \left[\frac{\partial^3 h_{jk}}{\partial t^3}\right] - \left[\frac{\partial^3 h_{00}}{\partial x^j\, \partial x^k\, \partial t}\right] + \left[\frac{\partial^3 h_{0j}}{\partial x^k\, \partial t^2}\right]\right) \\ - \left[\frac{\partial^3 h_{kn}}{\partial x^j\, \partial x^n\, \partial t}\right] + \left[\frac{\partial^3 h_{jk}}{\partial x^n\, \partial x^n\, \partial t}\right] + \left[\frac{\partial^3 h_{nn}}{\partial x^j\, \partial x^k\, \partial t}\right] - \left[\frac{\partial^3 h_{jn}}{\partial x^k\, \partial x^n\, \partial t}\right] = 0,\end{gathered} \tag{9.3}$$

in which $j, k, n = 1, 2, 3$ and there is a summation on the index n over this range.

Let us now substitute the expressions which are given by the kinematical compatibility conditions (8.15), (8.16) and (8.17) for the bracket quantities in (9.3). The direct result of this substitution is a rather lengthy set of equations which we shall not write down explicitly; however when we avail ourselves of the equations (2.7) and (2.8), the conditions

(5.2), (5.15), (5.17) and certain relations resulting from these conditions by differentiation, we find that the equations in question reduce to

$$2\frac{\delta C_{jk}}{\delta t} - 2G\Omega C_{jk} + Gg^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma}(C_{nj}\, x_{k\tau} + C_{nk}\, x_{j\tau})x_{n\beta}$$

$$- GC_{nj/\alpha}\, g^{\alpha\beta}(x_{k\beta}\, \nu_n + x_{n\beta}\, \nu_k) - GC_{nk/\alpha}\, g^{\alpha\beta}(x_{j\beta}\, \nu_n + x_{n\beta}\, \nu_j)$$

$$- E_{0j}\, \nu_k - E_{0\beta}\, \nu_j - GE_{nj}\, \nu_n\, \nu_k - GE_{nk}\, \nu_n\, \nu_j + \left(GE_{nn} - \frac{E_{00}}{G}\right)\nu_j\, \nu_k = 0,$$

in which the quantity Ω, defined by (2.11), is the mean curvature of the surface $\Sigma(t)$. We also find that

$$C_{jk}\frac{\delta C_{jk}}{\delta t} + GC_{jk}\, C_{nj}\, g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma}\, x_{k\tau}\, x_{n\beta}$$
$$= G\Omega C_{jk}\, C_{jk} + GC_{jk}\, g^{\alpha\beta}(C_{nj/\alpha}\, \nu_n)x_{k\beta}, \tag{9.4}$$

when we multiply the above relations by C_{jk} and sum on the repeated indices. But

$$C_{nj/\alpha}\, \nu_n = (C_{nj}\, \nu_n)_{/\alpha} - C_{nj}\, \nu_{n/\alpha} = C_{nj}\, g^{\sigma\tau} b_{\alpha\sigma}\, x_{n\tau},$$

by application of (2.10) and (5.17). Hence we have

$$\begin{aligned} g^{\alpha\beta}(C_{nj/\alpha}\, \nu_n)x_{k\beta} &= C_{nj}\, g^{\alpha\beta} g^{\sigma\tau} b_{\alpha\sigma}\, x_{k\beta}\, x_{n\tau} \\ &= C_{nj}\, g^{\alpha\tau} g^{\sigma\beta} b_{\alpha\sigma}\, x_{k\tau}\, x_{n\beta} \\ &= C_{nj}\, g^{\sigma\tau} g^{\alpha\beta} b_{\alpha\sigma}\, x_{k\tau}\, x_{n\beta}, \end{aligned}$$

when we make the indicated changes of indices. Thus we see that the second sets of terms in the left and right members of (9.4) are the same and hence cancel from this equation. Introducing the wave strength W, defined by the equation (5.14), we now find that (9.4) becomes

$$\frac{\delta W}{\delta t} = G\Omega W. \tag{9.5}$$

Finally the velocity G can be eliminated from (9.5) by means of the substitution

$$\frac{\delta W}{\delta t} = \frac{dW}{d\sigma}\frac{d\sigma}{dt} = G\frac{dW}{d\sigma},$$

in which use is made of the last equation (2.12). *The resulting equation, namely*

$$\frac{dW}{d\sigma} = \Omega W, \tag{9.6}$$

is the differential equation for the variation of the wave strength W along the normal trajectories to the wave surfaces. The purely geometrical nature of this problem is apparent from the equation (9.6).

X. Decay and Growth of Wave Strength During Propagation

In the case of a plane gravitational wave we have $\Omega = 0$ and hence we see from (9.6) that the strength of the wave remains constant during its propagation. For a spherical wave propagated outward, i.e. such that the radius R of the wave increases during the propagation, the mean curvature Ω is given by $\Omega = -1/R$; making this substitution for Ω in (9.6) and integrating the resulting equation, in which σ can evidently be replaced by the variable R, we obtain

$$W = W_0 \left(\frac{R_0}{R}\right),$$

where W_0 denotes the wave strength when the sphere has radius R_0. The strength of such a wave thus decreases to zero in accordance with this equations as R becomes indefinitely large.

In general [2]

$$\Omega = \frac{\Omega_0 - K_0 \sigma}{1 - 2\Omega_0 \sigma + K_0 \sigma^2}, \tag{10.1}$$

where Ω_0 and K_0 are the mean and Gaussian curvatures at a point P_0 on the surface $\Sigma(t_0)$ and Ω is the mean curvature of the surface $\Sigma(t)$ at a point P which lies a distance σ along the normal trajectory N joining P to P_0. Making the substitution (10.1) in (9.6) and integrating the equation so obtained, we have

$$W = \frac{W_0}{\sqrt{1 - 2\Omega_0 \sigma + K_0 \sigma^2}}, \tag{10.2}$$

as the explicit form of the equation for the variation of the wave strength W along the trajectory N. From our previous discussion of the equation (10.2) in connection with the variation of the strength of a longitudinal or a shear wave in an unstrained elastic medium [3] the following result can immediately be stated. *The strength W of a spatial gravitational* *wave, propagated into the space-time continuum whose element of distance ds is given by the form* (1.5), *will approach zero monotonically as the time of propagation increases indefinitely from an initial time t_0, if, and only if, the wave surface $\Sigma(t_0)$ is* (α) *a developable surface* (*not a plane*) *or* (β) *a surface of positive curvature and* (γ) *the direction of propagation of $\Sigma(t_0)$ is away from the centers of curvature at points of this surface.* The last requirement (γ), which is needed to insure the inequality $\Omega_0 < 0$ at points of the surface $\Sigma(t_0)$, can also be expressed by the assumption that $\Sigma(t_0)$ is convex in the sense that the straight line normals issuing from this surface in the direction of the propagation do not intersect and hence form a congruence of curves in the three dimensional Euclidean space.

When the above conditions are not met the strength of the wave may become infinite as a result of the propagation. For example if $\Sigma(t_0)$ is a spherical surface, propagated toward its center of curvature 0, the mean curvature will have the value $1/R$ where R is the radius of the moving wave surface $\Sigma(t)$ and thus we find that $W \to \infty$ as the wave converges to the point 0. Also if $\Sigma(t_0)$ is a circular cylinder whose direction of propagation is toward the central axis, it follows readily that W will become infinite along the axis of the cylinder. In general it is easily seen from the form of the equation (10.2) that points of infinite wave strength W may arise which will sweep out a two dimensional surface S in the Euclidean space during the course of the propagation. One would expect this surface S to revert to a wave associated with a *finite* but stronger discontinuity in order to circumvent the occurrence of infinite discontinuities in derivatives of the gravitational potentials.

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